MATHEMATICS

Stage 3C/3D

WACE Examination 2010

Final Marking Key

Calculator-free
and
Calculator-assumed

This 'stand alone' version of the WACE Examination 2010 Final Marking Key is provided on an interim basis.

The Standards Guide for this examination will include the examination questions, marking key, question statistics and annotated candidate responses. When the Standards Guide is published, this document will be removed from the website.

Section One: Calculator Free **(40 marks)**

Question 1 **(4 marks)**

Differentiate the following, without simplifying:

(a)

$$y = \frac{x-1}{x^2+4}$$

(2)

Solution
$\frac{dy}{dx} = \frac{(x^2+4)(1)-(x-1)2x}{(x^2+4)^2}$
Specific Behaviours
Applies quotient rule ✓ Differentiates numerator and denominator ✓ Answers which do not have the correct structure of the quotient rule will be awarded zero marks Simplification, correct or otherwise, is ignored

(b) $y = x^5 e^{-3x}$ (2)

Solution
$\frac{dy}{dx} = x^5(-3e^{-3x}) + 5x^4 e^{-3x}$
Specific Behaviours
Applies product rule ✓ Differentiates each factor correctly ✓ Answers which do not have the correct structure of the product rule will be awarded zero marks

Question 2 **(4 marks)**

Determine the domain and range of $f\big(g(x)\big)$, given that $f(x) = \sqrt{1-x}$ and $g(x) = 3^x - 8$

Solution
$f\big(g(x)\big) = \sqrt{1-g(x)} = \sqrt{1-(3^x-8)} = \sqrt{9-3^x}$. So $f\big(g(x)\big)$ is defined provided that $9-3^x \geq 0$, i.e. $x \leq 2$. So the domain is $(-\infty, 2]$ If $x \leq 2$ then $0 \leq \sqrt{9-3^x} \leq 3$ Now $f\big(g(2)\big) = \sqrt{9-3^2} = 0$ Since $3^x > 0$ and $3^x \to 0$ as $x \to -\infty$, $f\big(g(x)\big) \to \sqrt{9} = 3$ as $x \to -\infty$ So the range is $[0,3)$
Specific Behaviours
Obtains expression for $f(g(x))$ ✓ Obtains domain ✓ Obtains range limits 0 and 3 ✓ Includes 0 and excludes 3 ✓ Stating a range of $y \geq 0$ and y<3 to get both marks

Question 3 **(5 marks)**

Find the maximum and minimum values over the interval $1 \le x \le 5$ of the function

$$f(x) = 3x + \frac{16}{x^3}$$

Solution	
Since $f(x)$ is differentiable the extreme values occurs at the end points or at the critical points. $f'(x) = 3 - 3\frac{16}{x^4} = 3\left(1 - \frac{16}{x^4}\right) = 0$ when $x = 2$ $f(1) = 3 + 16 = 19$, $f(2) = 3 \times 2 + \frac{16}{2^3} = 6 + 2 = 8$, and $f(5) = 3 \times 5 + \frac{16}{5^3} = 15\frac{16}{125}$. So $f_{max} = 19$ and $f_{min} = 8$.	
Specific Behaviours	
Differentiates	✓
Finds x-value at critical point	✓
Evaluates y-value at critical point	✓
Checks values at end points and critical point	✓
States maxiumum and minimum values, clearly indicating that these are functional values, not x-values	✓

Question 4 **(3 marks)**

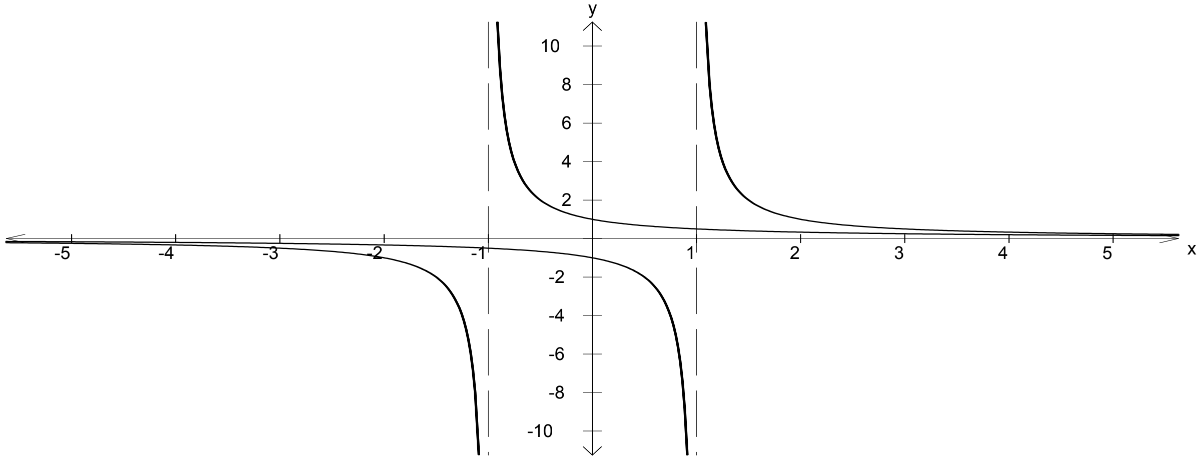
Solve for x the inequality

$$\frac{1}{x-1} < \frac{1}{x+1}$$

Solution	
$\frac{1}{x-1} < \frac{1}{x+1} \Leftrightarrow \frac{1}{x+1} - \frac{1}{x-1} > 0 \Leftrightarrow \frac{(x-1)-(x+1)}{x^2-1} = \frac{-2}{x^2-1} > 0 \Leftrightarrow x^2 - 1 < 0 \Leftrightarrow -1 < x < 1$	
Specific Behaviours	
Rearranges inequality with 0 on one side	✓
Simplifies	✓
Obtains answer	✓
Testing critical regions may also be a successful strategy	

Alternative solution

Sketches both hyperbolas ✓✓
Obtains interval $-1 < x < 1$✓



Question 5 **(6 marks)**

(a) Evaluate $\int_1^3 (x^3 - 1)dx$ (3)

Solution	
$\int_1^3 (x^3 - 1)dx = \left(\frac{1}{4}x^4 - x\right)\Big\vert_{x=1}^{x=3}$ $= \left(\frac{1}{4}3^4 - 3\right) - \left(\frac{1}{4} - 1\right) = \left(\frac{81}{4} - 3\right) - \left(-\frac{3}{4}\right) = \frac{84}{4} - 3 = 18$	
Specific Behaviours	
Integrates	✓
Evaluates at limits	✓
Simplifies	✓

(b) Determine $\int x(1 - x^2)^{10} dx$ (3)

Solution	
So $\int x(1 - x^2)^{10} dx = -\frac{1}{22}(1 - x^2)^{11} + c$	
Specific Behaviours	
Finds $(1 - x^2)^{11}$	✓
Calculates the correct coefficient ($-\frac{1}{22}$)	✓
Includes constant of integration	✓

Question 6 **(3 marks)**

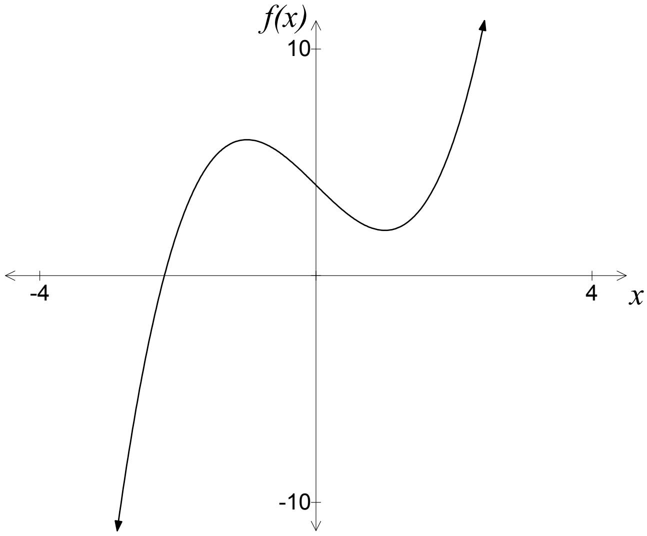
A certain type of computer password is 8 characters long. Six of the characters are lower-case letters from the English alphabet, i.e. members of the 26-element set $\{a, b, c, \ldots, x, y, z\}$. The other 2 characters are decimal digits. However, the decimal digits must occur consecutively. So $gyjp53iw$ is a possible password, but $af4tfz0y$ is not.

How many possible passwords are there? Give your answer as an arithmetical expression, without evaluating.

Solution	
26^6 ways of choosing the letters and 10^2 ways of choosing the digits There are 7 ways of choosing consecutive positions for the decimal digits. So by the multiplication principle there are $26^6 \times 10^2 \times 7$ possible passwords.	
Specific Behaviours	
Obtains number of letter choices	✓
Obtains number of digit choices	✓
Obtains number of possible positions for the digits	✓
Note that answers disallowing repetition will incur a one-mark deduction	

Question 7 **(10 marks)**

The graph of $y = f(x) = x^3 - 3x + 4$ is shown below.



(a) Determine the coordinates of the stationary points of the function f. (3)

Solution
$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 0$ when $x = \pm 1$ $f(1) = 1^3 - 3 \times 1 + 4 = 2$ and $f(-1) = (-1)(-1)^3 - 3 \times (-1) + 4 = 6$ So the coordinates of the stationary points are $(1,2)$ and $(-1,6)$
Specific Behaviours
Differentiates ✓ Obtains x coordinates of turning point ✓ Obtains y coordinates of turning point ✓

(b) For what values of x is it true that $f'(x) < 0$ and $f''(x) > 0$? (2)

Solution
$f'(x) = 3(x^2 - 1) < 0$ when $-1 < x < 1$ $f''(x) = 6x > 0$ when $x > 0$ So the solution is $0 < x < 1$
Specific Behaviours
Obtains inequality for f' ✓ Obtains inequality for f'' and solution ✓ One mark deducted if 0 and 1 are included in the solution interval No deduction if the two intervals are stated separately

(c) Without integrating, use the graph of $y = f(x)$ to explain why $\int_{-1}^{1} f(x)dx = 8$. (2)

Solution
The function is anti-symmetric about the point $(0,4)$. i.e. $x > 0,\ \ f(-x) - 4 = 4 - f(x)$ (algebraically) or, for $x > 0,\ \ f(-x)$ is just as much above 4 as $f(x)$ is below 4. (in words) So $\int_{-1}^{1} f(x)dx$ is equal to the area of a rectangle of width 2 and height 4, i.e. 8
Specific Behaviours
Recognises symmetry ✓ Obtains area of rectangle ✓ Various geometrical constructions eg. trapezium can be used correctly

The function $g(x)$ is defined by $g(x) = f(2x)$

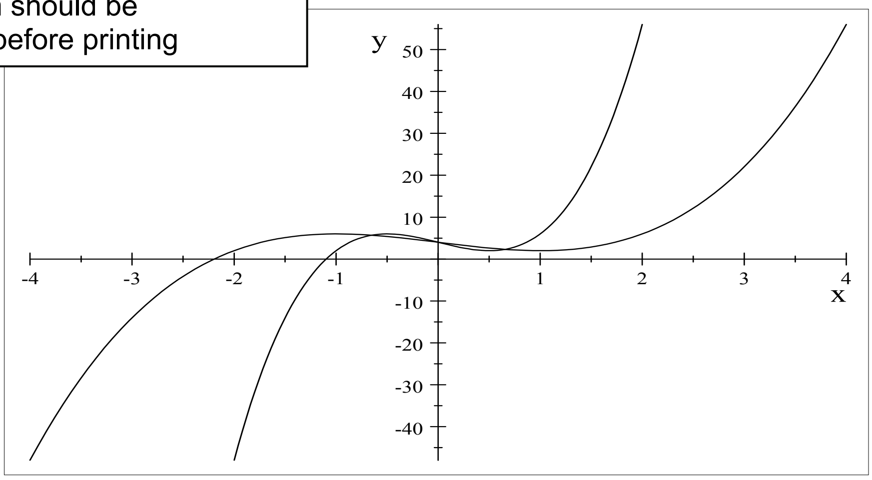
(d) Show that $g(x) = 8x^3 - 6x + 4$ (1)

Solution
$g(x) = f(2x) = (2x)^3 - 3(2x) + 4 = 8x^3 - 6x + 4$
Specific Behaviours
Expands formula for $f(2x)$ ✓

(e) Sketch on the axes on page 6 the graph of $y = 8x^3 - 6x + 4$ (2 marks)

Solution

This diagram should be reformatted before printing



Specific Behaviours
Displays contraction while y-intercept is unaffected ✓ Shows turning points at -0.5 and 0.5 with y-values unchanged ✓

Question 8 **(5 marks)**

Solve the system of equations
$$\begin{aligned} x - 4y - 3z &= 1 \\ x + 2y + 3z &= 4 \\ 3x - 8y - z &= 1 \end{aligned}$$

Solution
$\begin{aligned} x - 4y - 3z &= 1 \\ x + 2y + 3z &= 4 \\ 3x - 8y - z &= 1 \end{aligned} \Leftrightarrow \begin{aligned} x - 4y - 3z &= 1 \\ 6y + 6z &= 3 \text{ (eqn2-eqn1)} \\ 4y + 8z &= -2 \text{ (eqn3-3eqn1)} \end{aligned} \Leftrightarrow \begin{aligned} x - 4y - 3z &= 1 \\ y + z &= \tfrac{1}{2} \text{ (eqn2÷6)} \\ y + 2z &= -\tfrac{1}{2} \text{ (eqn3÷4)} \end{aligned}$
$\Leftrightarrow \begin{aligned} x - 4y - 3z &= 1 \\ y + z &= \tfrac{1}{2} \text{ (eqn2÷6)} \\ z &= -1 \text{ (eqn3 – eqn2)} \end{aligned}$ So $z = -1$.
Back substituting gives $y - 1 = \frac{1}{2}$, i.e. $y = \frac{3}{2}$, and $x - 4 \times \frac{3}{2} - 3 \times (-1) = 3$, i.e. $x = 4$

Specific Behaviours	
Eliminates (mark for each step)	✓✓✓
Back substitutes (for y and then for x)	✓✓

Section Two: Calculator assumed (80 marks)

Question 9 (7 marks)

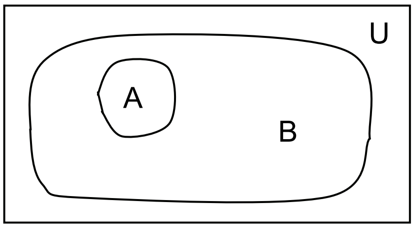
Suppose that $\mathrm{P(A)} = 0.5$ and that $\mathrm{P(A \cup B)} = 0.8$

(a) What is the maximum possible value of $\mathrm{P(A \cap B)}$? (2 marks)

Solution

Maximum value of $\mathrm{P(A \cap B)}$ occurs when A is a subset of B.

Then $\mathrm{P(A \cap B)} = \mathrm{P(A)} = 0.5$



Specific Behaviours

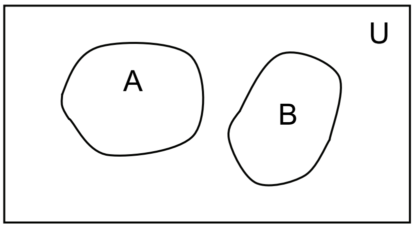
Recognises condition	✓
Obtains answer	✓

(b) What is the minimum possible value of $\mathrm{P(B)}$? (2 marks)

Solution

Minimum value of $\mathrm{P(A \cap B)}$ occurs when A and B are disjoint.

Then $\mathrm{P(B)} = \mathrm{P(A \cap B)} - \mathrm{P(A)} = 0.8 - 0.5 = 0.3$



Specific Behaviours

Recognises disjointness	✓
Obtains answer	✓

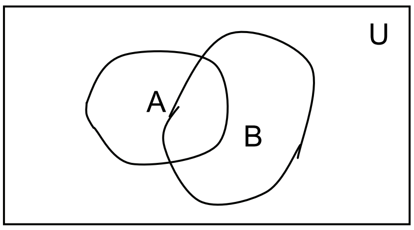
(c) What is the value of $\mathrm{P(B)}$ if A and B are independent? (3)

Solution

Let $x = \mathrm{P(B)}$. If A and B are independent, then $\mathrm{P(A \cap B)} = \mathrm{P(A)P(B)}$

Also $\mathrm{P(A \cup B)} = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A \cap B)}$

So $0.8 = 0.5 + x - 0.5x$
So $\mathrm{P(B)} = x = 0.6$



Specific Behaviours

Uses formula for $\mathrm{P(A \cap B)}$	✓
Uses formula for $\mathrm{P(A \cup B)}$	✓
Obtains answer	✓

Use of the formulas requires that values be substituted. Stating formulas is insufficient

Question 10 **(6 marks)**

Helium gas is being pumped into a balloon. The balloon maintains a spherical shape as it inflates, and its volume increases at a constant rate of 600 cc per minute.

(a) At what rate is the radius of the balloon increasing at the moment when the volume of the balloon is 20 litres? (1 litre = 1000 cc.) (4)

Solution	
$V = \frac{4}{3}\pi r^3$, and so $\frac{dV}{dr} = 4\pi r^2$(*), and hence $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$ Now $\frac{dV}{dt} = 600$. When $V = 20000$, $r = 16.84$ (from calculator) So $600 = 4 \times \pi \times 16.84^2 \times \frac{dr}{dt}$, and therefore $\frac{dr}{dt} = 0.1684$ (from calculator) So the radius is increasing at the rate of 0.1684 cm per minute.	
Specific Behaviours	
Obtains equation (*)	✓
Finds $\frac{dV}{dt}$	✓
Evaluates r at V = 20 L	✓
Obtains answer	✓

(b) Use the formula $\delta y \approx \frac{dy}{dx}\delta x$ to estimate the amount by which the radius will increase in the next second. (2)

Solution	
$\delta r \approx \frac{dr}{dt}\delta t$ and $\delta t = 1/60$ So $\delta r \approx 0.1684 \times \frac{1}{60} = 0.0028$ So the radius will increase by approximately 0.0028 cm.	
Specific Behaviours	
Uses $\frac{dr}{dt}$ value found in (a)	✓
Uses correct value of δt	✓
Alternative reasoning using $\delta r \approx \frac{dr}{dV}\delta V$ is possible	

Question 11 **(6 marks)**

A radioactive substance is decaying exponentially, according to the formula

$$A(t) = A(0)e^{-kt}, \quad \text{where } A(t) \text{ kg is the amount at time } t \text{ years.}$$

(a) Determine k, correct to 4 significant figures, given that the half-life of the substance is 12 years. (2)

Solution	
Since $A(12) = \frac{1}{2}A(0)$, $\frac{1}{2} = e^{-12k}$ So $k = 0.05776$ (from calculator)	
Specific Behaviours	
Obtains answer	✓
Gives 4 significant figures	✓

A second radioactive substance is also decaying exponentially, according to the formula

$$B(t) = B(0)e^{-0.04t}, \text{ where } B(t) \text{ kg is the amount at time } t \text{ years.}$$

(b) Which of these substances is decaying faster? Justify your answer briefly. (1)

Solution	
A is decaying faster because $k > 0.04$	
Specific Behaviours	
Obtains answer with correct reasoning	✓

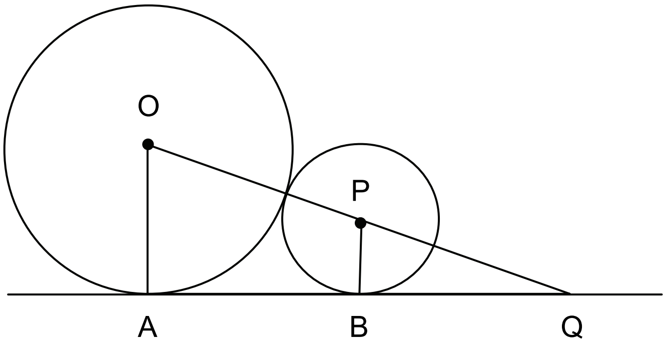
At a certain location there was exactly the same amount of these two substances at the beginning of the year 2010.

(c) In what year will the ratio of the amount of one of these substances to the other be 2:1?(3)

Solution	
Since $A(t) = B(t)/2$, $A(0)e^{-0.05776t} = B(0)e^{-0.04t}/2$ (*) But $A(0) = B(0)$, so $e^{-0.05776t} = e^{-0.04t}/2$, i.e. $\frac{e^{-0.04t}}{e^{-0.05776t}} = 2$ i.e. $\frac{e^{-0.04t}}{e^{-0.05776t}} = e^{0.05776t-0.04t} = 2$ So $t = 39.03$ (from calculator) So the year will be 2049	
Specific Behaviours	
Obtains equation (*)	✓
Obtains value of t	✓
Interprets answer to obtain year	✓

Question 12 **(6 marks)**

Two circles are tangent to a line and to each other, as shown in the diagram below. The radius of the larger circle is twice the radius of the smaller circle.



(a) Prove that the triangles AOQ and BPQ are similar. (2)

Solution	
$\angle OAQ = \angle PBQ = 90°$ and $\angle OQA = \angle PQB$ (common) So the triangles have equal angles, and so are similar	
Specific Behaviours	
Recognises matching angles	✓
Recognises that equal angles imply similarity	✓
Informal reasoning accepted	
Extraneous information ignored	

(b) Show that $PQ = 3r$ where r is the radius of the smaller circle. (2)

Solution	
Let the radius of the smaller circle be r cm. Then $PB = r$, $OA = 2r$ and $OP = 3r$ (*) By similarity $\frac{OQ}{PQ} = \frac{OA}{PB} = \frac{2r}{r} = 2$. So $OQ = 2PQ$ and hence $PQ = OP = 3r$	
Specific Behaviours	
Obtains an expression for OP	✓
Reasons correctly that PQ = OP	✓

(c) Find the radius of the smaller circle, given that AB = 20 cm. (2)

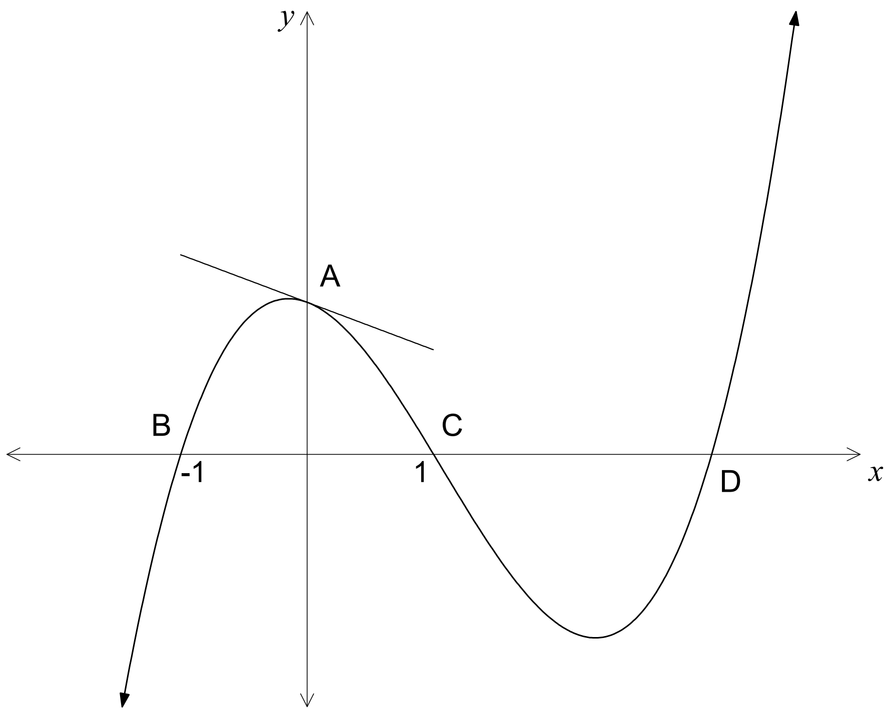
Solution	
By similarity $\frac{AQ}{BQ} = \frac{OA}{PB} = \frac{2r}{r} = 2$. So $BQ = AB = 20$ By Pythagoras' theorem, $PQ^2 = PB^2 + BQ^2$, i.e. $(3r)^2 = r^2 + (20)^2$(*) So $8r^2 = 400$, and hence $r = 5\sqrt{2} = 7.071$ (to 4 decimal places) So the radius of the smaller circle is $5\sqrt{2}$ = 7.071 cm	
Specific Behaviours	
Uses Pythagoras' theorem	✓
Obtains answer	✓

Question 13 **(5 marks)**

The diagram below shows part of the graph of $y = (x^2 - 1)(x - d)$ where $a > 0$ and $d > 1$

The graph intercepts the *y*-axis at the point A. The graph intercepts the *x*-axis at the point B where $x = -1$, the point C where $x = 1$, and at the point D.

The diagram also shows part of the tangent to the graph at the point A.

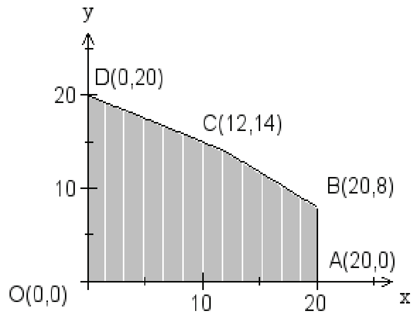


Show that the tangent at A intersects the *x*-axis at D.

Solution	
$y = (x^2 - 1)(x - d) = x^3 - dx^2 - x + d$ So $\frac{dy}{dx} = 3x^2 - 2dx - 1 = -1$ when $x = 0$. Also $y = d$ when $x = 0$ So the equation of the tangent at A is $y - d = -1(x - 0)$ i.e. $y = d - x$ Solving $y = d - x = 0$ gives $x = d$. So the tangent intercepts the x – axis at $(d, 0)$. But $y = 0$ when $x = d$. So D has coordinates $(d, 0)$. So the tangent and the curve intercept the x – axis at the same point D.	
Specific Behaviours	
Differentiates	✓
Evaluates $y'(0)$	✓
Obtains equation of the tangent	✓
Recognises that the curve meets the x – axis at $(d, 0)$	✓
Recognises that the tangent meets the x – axis at $(d, 0)$	✓

Question 14 **(6 marks)**

The feasible region of a linear programming problem is shown below.



The objective function is $P = 60x + 100y$

(a) Find the maximum value of P in the feasible region. (2)

Solution	
The maximum value occurs at one of the corner points At O $P = 0$, at A $P = 60 \times 20 = 1200$, at B $P = 60 \times 20 + 100 \times 8 = 2000$, at C $P = 60 \times 12 + 100 \times 14 = 2120$, and at D $P = 100 \times 20 = 2000$. So $P_{max} = 2120$.	
Specific Behaviours	
Evaluates corners	✓
Obtains answer	✓

(b) Now suppose that the objective function changes to $P = cx + 100y$, where $c > 60$.

What is the maximum possible value of the constant c, given that the maximum value of P still occurs at the same corner point? (2)

Solution	
As c increases the lines of constant P values become steeper, and eventually the optimal solution will move from C to B. The changeover occurs when the line $cx + 100y = k$ has the same slope as the chord BC, i.e. when $-\frac{c}{100} = \frac{14-8}{12-20} = -\frac{6}{8}$, i.e. when $c = 75$ OR Value of P at (12,14) = value at (20,8) 12c + 1400 = 20c + 800	
Specific Behaviours	
Recognises the critical slope or equates P values at appropriate endpoints	✓
Obtains answer	✓

(c) Now suppose that the additional constraint $x + y \leq 27$ is imposed. Does this change the maximum value of P? Justify your answer. (2)

Solution	
Answer: No. Reason: The point C satisfies the additional constraint, since $12 + 14 \leq 27$. So C will still be in the feasible region. So the maximum value of P will still occur at C, and hence is unchanged.	
Specific Behaviours	
Obtains answer	✓
Reasons correctly	✓
Answers which state that the new constraint does not intersect the feasible region will be awarded one mark only.	

Question 15 **(6 marks)**

For some positive integers n the decimal expansion of $1/n$ is finite, and for others it is infinite.

For example, the decimal expansion of $1/25$ is finite since $1/25 = 0.04$, whereas the decimal expansion of $1/24$ is infinite since $1/24 = 0.041666 \cdots = 0.041\dot{6}$.

(a) Write down the decimal expansions of $1/n$ for some small values of n, including 6, 8, 11, and 20. (1)

Solution
$1/6 = 0.1666 \ldots = 0.1\dot{6}$, $1/8 = 0.125$ $1/11 = 0.090909 \ldots = 0.\dot{0}\dot{9}$ and $1/20 = 0.05$
Specific Behaviours
Obtains all four answers with no rounding of recurring decimals ✓

(b) Write down a conjecture about the prime factors of n if the decimal expansion of $1/n$ is finite.
Hint: you may wish to evaluate the decimal expansion of $1/n$, for other positive integers n, until you notice the pattern. (2)

Solution
The prime factors of n are either 2 or 5.
Specific Behaviours
Conjectures correctly about powers of two ✓ Conjectures correctly about powers of five ✓

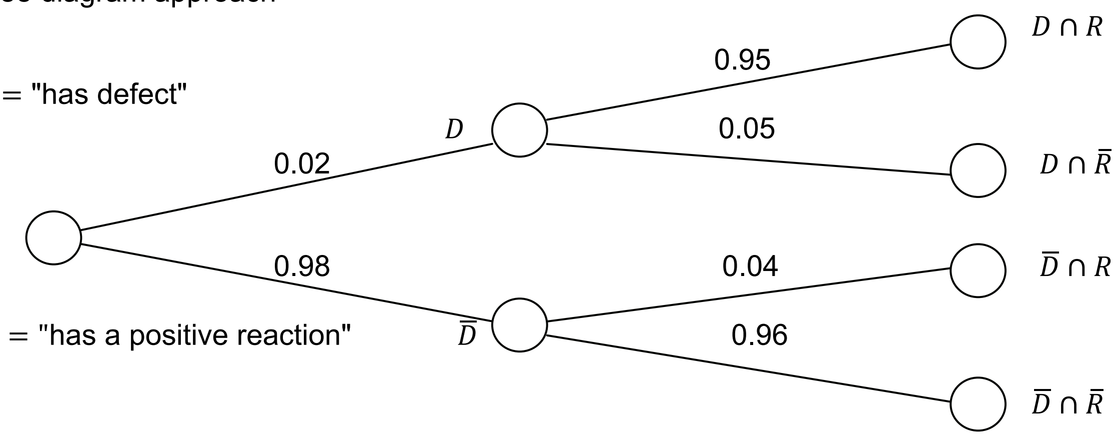
(c) Prove the claim that you made in part (b). (3)

Solution
If the decimal expansion of $1/n$ is finite then $\frac{1}{n} = \frac{A}{10^k}$ for some positive integers A and k. (*) Then $n = \frac{10^k}{A}$, and so n is a factor of 10^k. (**) So the prime factors of n must also be prime factors of 10, i.e. either 2 or 5.
Specific Behaviours
Obtains equation (*) or otherwise indicates an understanding of the role of powers of 10 in the proof ✓ Draws conclusion (**) ✓ Completes proof ✓

Question 16 **(5 marks)**

A new drug has been developed that can be used to test whether a person has a certain type of genetic defect. However the test is not perfect: only 95% of people with the defect have a positive reaction to the drug, and 4% of people without the defect have a positive reaction. It is also known that 2% of the total population has the genetic defect.

(a) What is the probability that a person chosen at random will have a positive reaction to the drug? (3)

Solution
Tree diagram approach
$D =$ "has defect"
$R =$ "has a positive reaction"

$P(R) = P(D \cap R) + P(\overline{D} \cap R) = 0.02 \times 0.95 + 0.98 \times 0.04 = 0.0582$
Specific Behaviours
Draws tree diagram with edge probabilities ✓✓
Obtains answer ✓

(b) What proportion of the people who have a positive reaction actually have the genetic defect? (2)

Solution
$P(D\|R) = \dfrac{P(D \cap R)}{P(R)} = \dfrac{0.02 \times 0.95}{0.0582} = 0.3265$
So (only) 32.6% of those who test positive actually have the genetic defect.
Specific Behaviours
Uses conditional probability formula ✓
Obtains answer ✓

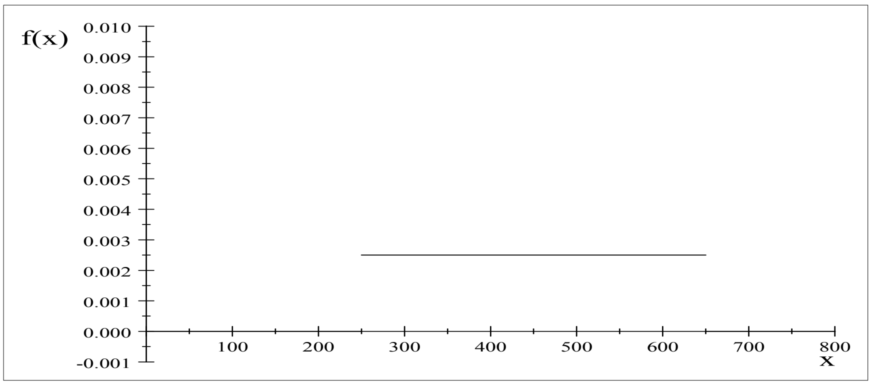
Question 17 **(11 marks)**

The cost, \$C, of insuring a house in a particular town is a uniformly distributed random variable, with minimum and maximum values \$250 and \$650 respectively.

The average cost is \$450 and the standard deviation is \$115.47.

(a) Sketch the graph of the density function of C. (2)

Solution



Specific Behaviours	
Draws horizontal line with height 0.0025	✓
Displays limits 250 and 650	✓

(b) What is the probability that a randomly-chosen house in the town costs more than \$500 to insure? (2)

Solution	
$P(C > 500) = \frac{650 - 500}{400} = \frac{3}{8} = 0.375$	
Specific Behaviours	
Displays specific formula for $P(C)$	✓
Evaluates	✓

(c) What is the probability that exactly 2 of 5 randomly-chosen houses in the town cost more than \$500 each to insure? (2)

Solution	
Let X denote the number that cost more than \$500 Then $X \sim \text{Binomial}(n = 5, p = 0.375)$ So $P(X = 2)$ =0.343 (from calculator)	
Specific Behaviours	
Recognises binomial random variable with correct parameters	✓
Obtains answer	✓

The total cost of insurance for 25 houses in the town owned by a real estate syndicate is \$12 500. The syndicate suspects that this is unusually high.

(d) What is the average insurance cost for the houses owned by the syndicate? (1)

Solution	
$\bar{C} = \frac{12500}{25} = 500$ i.e. average cost is \$500	
Specific Behaviours	
Obtains answer	✓

(e) Use the Central Limit Theorem to estimate the probability that the total cost of insuring 25 randomly-chosen houses in the town will be at least \$12 500. (4)

Solution	
Total cost > \$12 500 corresponds to average cost > \$500. Let $\bar{C}$ the average cost (in dollars) of insuring 25 randomly-chosen homes. Then $\bar{C}$ is a random variable with mean 450 and standard deviation is $\frac{115.47}{\sqrt{25}} = 23.094$ Furthermore the distribution of $\bar{C}$ is approximately normal. So $P(\bar{C} > 500) \cong P\left(Z > \frac{500-450}{23.094}\right) = P(Z > 2.165) = 0.0152$ Thus the required probability is approximately 0.015	
Specific Behaviours	
Uses mean of $\bar{C}$	✓
Uses standard deviation of $\bar{C}$	✓
Identifies normal distribution	✓
Obtains answer	✓

Question 18 **(10 marks)**

The burn time, T seconds, of a randomly-chosen match produced by the Ever-Flame company is normally distributed, with a mean of 12.2 seconds and a standard deviation of 2.5 seconds.

(a) Calculate $P(T > 16)$ (1)

Solution	
$P(T > 16) = P\left(Z > \frac{16 - 12.2}{2.5}\right) = P(Z > 1.52) = 0.0643$	
Specific Behaviours	
Obtains answer	✓

(b) Find the value of *k,* given that 90% of the matches burn for longer than *k* seconds. (2)

Solution	
$P(T > k) = P\left(Z > \frac{k-12.2}{2.5}\right) = 0.90$ (*) So $\frac{k-12.2}{2.5} = -1.28$. Therefore $k = 9.00$	
Specific Behaviours	
Obtains equation (*)	✓
Obtains answer	✓

(c) If 10 matches are burned, find the probability that at least half burn for less than k seconds.
(2)

Solution	
Let X denote the number that burn for less than k seconds Then $X \sim \text{Binomial}(n = 10, p = 0.1)$ So $P(X \geq 5) = 0.0016$ (from calculator)	
Specific Behaviours	
Recognises binomial variable	✓
Obtains answer	✓

(d) Every week the company tests its matches by measuring the burn times of 1000 randomly-chosen matches.

What is the probability that the average burn time of the matches in such a sample will be between 12.15 and 12.25 seconds? (2)

Solution	
Let $\bar{X}$ denote the average burn time from a random sample of 1000 matches. Then $\bar{X}$ is a random variable with mean 12.2 and standard deviation is $\frac{2.5}{\sqrt{1000}} = 0.079$ Furthermore the distribution of $\bar{X}$ is normal. So $P(12.15 \leq \bar{X} \leq 12.25) = P\left(\frac{12.15-12.2}{0.079} \leq Z \leq \frac{12.25-12.2}{0.079}\right) = P(-0.63 \leq Z \leq 0.63)$ $= 0.7357 - 0.2643 = 0.4714$ (or straight from calculator) Thus the required probability is approximately 0.4714	
Specific Behaviours	
Obtains mean and standard deviation for $\bar{X}$	✓
Obtains answer	✓

(e) The rival Sure-Fire company produces matches whose burn times have the same standard deviation as the Ever-Flame matches, but whose mean, μ seconds, is unknown. Scientists plan to estimate μ using the average burn time of matches in a random sample of Sure-Fire matches.

How large should this sample be, if the scientists are to be 95% confident that this estimate will be correct to within 0.1 seconds? (3)

Solution	
The half-width of the 95% confidence interval is $1.96\frac{\sigma}{\sqrt{n}} = \frac{1.96 \times 2.5}{\sqrt{n}} = \frac{4.9}{\sqrt{n}}$. If $\frac{4.9}{\sqrt{n}} < 0.1$, then $n > \left(\frac{4.9}{0.1}\right)^2 = 2401$ So the sample size should be at least 2041.	
Specific Behaviours	
Recognises that half width is the maximum error	✓
Uses formula for half width	✓
Obtains answer	✓

Question 19 **(7 marks)**

The acceleration, $a(t)$ m s^{-2}, of an object moving in a straight line is given by

$$a(t) = At + B, \text{ where } A \text{ and } B \text{ are non-zero constants.}$$

The object is at rest initially and again after 10 seconds, and the object returns to its initial position after T seconds.

(a) Evaluate T (4)

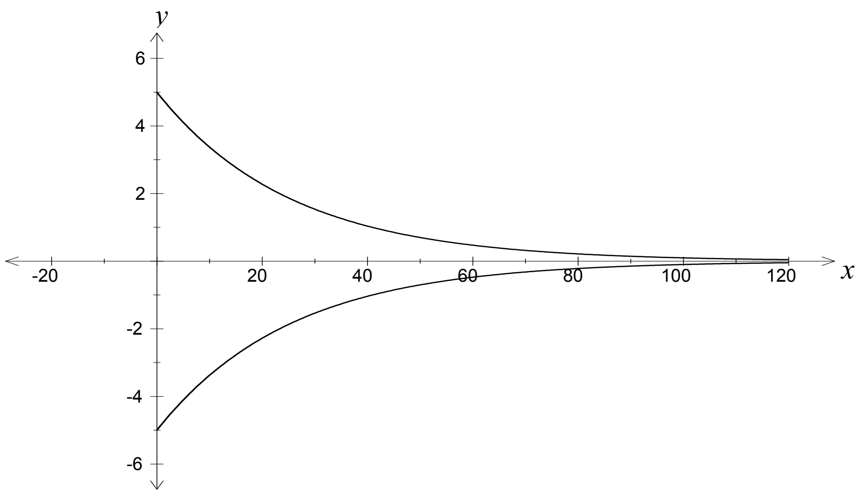
Solution
Integrating $a(t)$ gives $v(t) = \frac{1}{2}At^2 + Bt + C = \frac{1}{2}At^2 + Bt$ since $v(0) = 0$. (*)
Since $v(10) = 0, 50A + 10B = 0$, i.e. $B = -5A$ (**)
Integrating $v(t)$ gives $x(t) = \frac{1}{6}At^3 + \frac{1}{2}Bt^2 + D$
Since $x(T) = x(0) = D, \frac{1}{6}AT^3 + \frac{1}{2}BT^2 = 0,$ (***)
i.e. $\frac{1}{6}AT^3 - \frac{5}{2}AT^2 = \frac{1}{6}AT^2(T - 15) = 0$
Since $A \neq 0$ and $T \neq 0$, it follows that $T = 15$
Specific Behaviours
Obtains equation (*) ✓
Obtains equation (**) ✓
Obtains equation (***) or otherwise uses displacement to correctly generate an equation ✓
Obtains answer ✓

(b) Evaluate A and B, given that the acceleration is positive initially and that the object travels a distance of 1 kilometre in the first T seconds. (3)

Solution
The objects starts at one point, moves forward for 10 seconds and then returns to its starting point after 15 seconds.
So it travels 500 metres in the first 10 seconds. (*)
So $x(10) - x(0) = \frac{1}{6}A(10)^3 + \frac{1}{2}B(10)^2 = 500$
i.e. $\frac{1000}{6}A - 250A = 500,$
i.e. $A = -6$, and hence $B = 30$
Specific Behaviours
Obtains equation (*) ✓
Correctly evaluates A ✓
Correctly evaluates B ✓

Question 20 **(6 marks)**

The outline of a circularly symmetric tunnel, whose length is 120 metres, is shown below.



The top of the tunnel fits the curve $y = 5e^{-kx}$, where k is a constant, x and y are distances measured in metres, and $0 \le x \le 120$.

The total volume of the tunnel is 1000 cubic metres.

(a) Evaluate k correct to 3 significant figures. (3)

Solution
$V = \int_0^{120} \pi y^2 dx = \int_0^{120} \pi\left(5e^{-kx}\right)^2 dx = \int_0^{120} 25\pi e^{-2kx} dx$ $= 25\pi\left(-\frac{1}{2k}e^{-2kx}\right)\Big\vert_{x=0}^{x=120} = \frac{25\pi}{2k}\left(1 - e^{-240k}\right)$ Solving $\frac{25\pi}{2k}\left(1 - e^{-240k}\right) = 1000$ gives $k = 0.0393$ to 3 significant figures.
Specific Behaviours
Displays integral formula for V ✓ Evaluates k correctly ✓ Rounds k to 3sf ✓ An incorrect answer containing 3sf will be awarded no marks

(b) Could a man fit through the far end of the tunnel? Could a mouse? Justify your answers. (2)

Solution
$r = Ae^{-kt} = 5e^{-0.0393x} = 5e^{-0.0393\times 120} = 0.04475$ So the radius at the end of the tunnel is approximately 4.5 cm. A man could not fit through it but a mouse could.
Specific Behaviours
Obtains radius ✓ Draws correct conclusions ✓